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Classical symmetries of monopole by group theoretic methods

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Abstract

We use group theoretic methods to obtain the extended Lie point symmetries of the equations of motion for a charged particle in the field of a monopole. Cases with certain model magnetic fields and potentials are also studied. Our analysis gives the generators and Lie algebras generating the inherent symmetries. The equations of motion of a scalar particle probing the near horizon structure of a black hole is also treated likewise. We have also found the generators of Krause's complete symmetry groups for some of the above examples.

1 Introduction

In certain physical problems there may exist extra hidden symmetries which are not apparent, unless searched for. Some of the examples from classical considerations are, a particle in a $\frac{1}{r^2}$ potential in one dimension [1], the conserved Runge-Lenz vector of the Kepler problem and the extra symmetries of a charge moving in the field of a magnetic monopole[3, 4], and the generators beyond the Poincare invariance that give rise to conformal invariance in electrodynamics as well as in Yang-Mills theory. The existence of symmetries help in classifying and obtaining energy levels and eigenstates in quantum mechanical problems, generating new solutions and also formulating conservation laws. As we know manifestation of scale invariance in deep inelastic scattering had deep significance in the development gauge theories. The invariance under scale and conformal transformations also motivated the construction of a simple classical model which leads to conformal quantum mechanics[1]. Recently there have been a revival of interest in this model. This is due to the observation in string theory dynamics that a particle near a black hole possesses $SO(2,1)$ symmetries as in conformal quantum mechanics.[2].

The symmetries of charged particle - monopole system and the conformal quantum mechanics were obtained from physical reasonings and scale invariance. However, there exists a general programme to obtain the symmetries of the equations of motion of any such system by using the group theoretic methods of Lie. In this paper we use this method to find the Lie point symmetries of the monopole system as well as some other physically motivated systems.

A knowledge of the symmetry group of a system of differential equations leads to several types of applications[7]. For example, the symmetry group is helpful in finding solutions to the set of equations as well as constructing new solutions to the systems from the known ones. There have been efforts to generalise and extend the Lie approach by considering non-standard symmetries, as well. This helps in getting a wider class of solutions. Non-classical symmetries result by the weakening of invariance requirements of the differential equation under the symmetry transformations[8]. Another approach is to enlarge the space of independent variables by adding auxiliary variables and finding the symmetries. Then the symmetries related to the original system are figured out and the corresponding symmetries are called non-local[9, 10, 11]. This way, it also provides a method to classify

different classes of solutions corresponding to different symmetries. One can use, on the other hand, symmetry groups to classify families of differential equations depending on arbitrary parameters or functions. As an extension of this idea, Krause has introduced the important idea of complete symmetry of a differential equation so as to expand the symmetry group such that the manifold of solutions is an homogeneous space of the group and the group is specific to the system, i.e., no other system admits that symmetry group[12]. The complete symmetry group of the system is the group represented by the set of symmetries required to specify completely a system and its point symmetries[13, 14, 15, 16].

Further, just the enumeration of the symmetry generators sometimes provide much physical insight and quantitative physical results for which the full solutions are not required. The derivation of Kepler's third law of planetary motion and Runge-Lenz vectors[17], calculation of energy levels for hydrogen like atoms and generalized Kepler's problems, harmonic oscillators, Morse potentials[18], electron in a specific nonuniform magnetic field[19], being some such examples. In these the energy eigenvalues are obtained through the method of spectrum generating algebras which gives the Casimir invariants directly without explicit recourse to the solutions. Of course, the solutions are also obtained from the representation theory. For finding the continuous symmetries, Lie's method of group analysis seems to be the most powerful technique available. For example, Witten[20] had considered an example of the equation of motion of a particle in three dimensions constrained to move on the surface of a sphere in the presence of a magnetic monopole. This is the classical analogue of the Wess-Zumino model[21]. The equations of motion of such a system in the presence of a magnetic monopole cannot be obtained from the usual Lagrangian formulation unless one goes to a higher dimension. Hence, the usual method of finding the symmetries through Noether's theorem would have difficulties. So, to look for the continuous symmetries associated with such classical systems one has to analyze directly the equations of motion[22]. Similar is the case for Korteweg-de Vries equation which is not amenable to a direct Lagrangian formulation when expressed as a lowest order equation[23]. Another example is the Lorenz system of equations which have been dealt in the papers by Sen and Tabor[24], and Nucci[14]. For the classical systems, this procedure of finding directly the symmetries from equation of motion is, in some sense, more fundamental. This is because in certain cases many different Lagrangians may give rise to the same equations of motion. The group analysis of the equations of motion gives all the Lie point

group symmetry generators. In the cases where a Lagrangian formulation is possible, the usual Noether symmetries are a subset of the above generators. This subset of generators acting on the Lagrangian gives zero[17]. However, besides these there may be other generators obtained through group analysis which have direct physical significance, but not explicitly available from the consideration of usual Noether symmetries alone[25]. The reproduction of Kepler's third law in the planetary motion problem is such an example. The extension of this idea to the notion of Lie dynamical symmetries contains similarly a subclass known as Cartan symmetries. The Runge-Lenz vector can be obtained from such considerations. These symmetries are, further, related to the Lie-Bäcklund symmetries.

The application of these types of analysis to nonlocal cases have been widely studied through Bäcklund transformations and related techniques in the context of integrable systems containing infinite number of conservation laws[27]. The Thirring model has been analyzed by Morris [28]. The differential geometric forms developed earlier are used in the above analysis to obtain the prolongation structure[29]. Some other applications of these ideas to important problems from physics is comprehensively covered by Gaeta[30].

In this paper, as a first step towards a complete group analysis, we find the Lie point symmetries of the coupled set of differential equations representing the motion of a charged particle in three dimensions:

- (i) in the presence of a magnetic field proportional to the coordinate vector,
- (ii) in the presence of a magnetic monopole stationed at the centre of a sphere, with the constraint that the particle moves on the surface of this sphere of unit radius,
- (iii) in the presence of a $\frac{1}{r^2}$ potential
- (iv) in the presence of a magnetic monopole,
- (v) in the presence of a dyon,
- (vi) in a model magnetic field of the form $\mathbf{B} = \{0, 0, -\frac{\mathcal{B}}{x^2}\}$. Here \mathcal{B} is a constant,
- and
- (vii) a particle in a type of velocity dependent potential,

We also obtain the generators of the complete symmetry group of Krause for most of the above examples in this paper.

It should be noted that in the context of the symmetries of Wess-Zumino-Witten models, the symmetries in the higher dimension play by far the most important role and these have been fruitfully exploited [41].

Section 2 provides the outline of the method for group analysis of the equations of motion. In section 3 we use this method to find the generators of the point symmetries for the above examples. The symmetries of the equation of a scalar quantum particle near the horizon of a massive blackhole is considered in section 4. Next we follow the method of reduction of order introduced by Nucci[13, 31] to obtain the complete Krause symmetries for four of the above examples which is the content of section 5. Finally, Section 6 is devoted to physical interpretations and conclusions.

2 Symmetry Conditions

Typically we are interested in the coupled nonlinear set of equations representing the equations of motion of a particle in three dimensions. These are of the form

$$\ddot{x}_a = \beta \omega_a(x_i, \dot{x}_i, t) \quad (1)$$

where a dot represents derivative with respect to time, $a, i = 1, 2$, and 3 , and β is a constant involving mass, coupling constant etc.. The expressions for the function ω_a will be given explicitly for each example.

These set of equations can be analyzed by means of one parameter groups by infinitesimal transformations. We demand the equation to be invariant under infinitesimal changes of the explicit variable t , as well as simultaneous infinitesimal changes of the dependent functions x_a in the following way,

$$\begin{aligned} t \rightarrow \tilde{t} &= t + \epsilon \tau(t, x_1, x_2, x_3) + O(\epsilon^2), \\ x_a \rightarrow \tilde{x}_a &= x_a + \epsilon \eta_a(t, x_i) + O(\epsilon^2). \end{aligned} \quad (2)$$

Under $t \rightarrow \tilde{t}$ and $x_a \rightarrow \tilde{x}$, the equation changes to,

$$\ddot{\tilde{x}}_a = \beta \tilde{\omega}_a(\tilde{t}, \tilde{x}_i, \dot{\tilde{x}}_i) \quad (3)$$

To illustrate the procedure consider the simple case in one space dimension. We express the above equation in terms of t and q by using the transformation (2). Then the invariance condition implies that an expression containing various partial derivatives of τ and η is obtained which equates to zero. For example, we get

$$\frac{d\tilde{x}}{d\tilde{t}} = \frac{dx + \epsilon(\frac{\partial \eta}{\partial t} dt + \frac{\partial \eta}{\partial x} dx)}{dt + \epsilon(\frac{\partial \tau}{\partial t} dt + \frac{\partial \tau}{\partial x} dx)} + O(\epsilon^2) \quad (4)$$

and now relate the left hand side with $\frac{dx}{dt}$ by using binomial theorem for the denominator to obtain

$$\frac{d\tilde{x}}{d\tilde{t}} = \frac{dx}{dt} + \epsilon \left[\left(\frac{\partial \eta}{\partial x} - \frac{\partial \tau}{\partial t} \right) \frac{dx}{dt} - \frac{\partial \tau}{\partial x} \left(\frac{\partial x}{\partial t} \right)^2 \right] + O(\epsilon^2). \quad (5)$$

A similar procedure is followed to express $\frac{d^2 \tilde{x}}{d\tilde{t}^2}$ likewise. By substituting equations (2) - (5) for a given explicit expression for ω_a and remembering that $\frac{d^2 x_a}{dt^2} - \omega_a$ is zero, we obtain the desired partial differential equation whose solution would determine $\tau(t, x)$ and $\eta(t, x)$. In our case, of course, we have to find $\tau(t, x_1, x_2, x_3)$ and $\eta_a(t, x_1, x_2, x_3)$'s.

To relate these to the generators of the infinitesimal transformations we write

$$\tilde{t}(t, x_i; \epsilon) = t + \epsilon \tau(t, x_i) + \dots = t + \epsilon \mathbf{X} t + \dots \quad (6)$$

$$\tilde{x}_a(t, x_i; \epsilon) = x_a + \epsilon \eta(t, x_i) + \dots = x_a + \epsilon \mathbf{X} x_a + \dots \quad (7)$$

where the functions τ and η_a are components of tangent vectors at the points \tilde{t} and \tilde{x}_a defined by

$$\tau(t, x_i) = \left. \frac{\partial \tilde{t}}{\partial \epsilon} \right|_{\epsilon=0}, \quad (8)$$

$$\eta_a(t, x_i) = \left. \frac{\partial \tilde{x}_a}{\partial \epsilon} \right|_{\epsilon=0} \quad (9)$$

and the operator \mathbf{X} is given by

$$\mathbf{X} = \tau(t, x_i) \frac{\partial}{\partial t} + \eta_a(t, x_i) \frac{\partial}{\partial x_a}. \quad (10)$$

where repeated indices are summed. Following Stephani [17], we will find the infinitesimal generators of the symmetry under which the system of differential equations do not change. The symmetry is generated by \mathbf{X} and its extension

$$\dot{\mathbf{X}} = \tau \frac{\partial}{\partial t} + \eta_a \frac{\partial}{\partial x_a} + \dot{\eta}_a \frac{\partial}{\partial \dot{x}_a} \quad (11)$$

and the symmetry condition under transformations represented by equation (2) determines $\dot{\eta}_a$. In the expanded form the symmetry condition becomes

$$\begin{aligned} \eta_b \omega_{a,b} + (\eta_{b,t} + \dot{x}_c \eta_{b,c} - \dot{x}_b \tau_{,t} - \dot{x}_b \dot{x}_c \tau_{,c}) \frac{\partial \omega_a}{\partial \dot{x}_b} + \tau \omega_{a,t} + 2\omega_a (\tau_{,t} + \dot{x}_b \tau_{,b}) \\ + \omega_b (\dot{x}_a \tau_{,b} - \eta_{a,b}) + \dot{x}_a \dot{x}_b \dot{x}_c \tau_{,bc} + \dot{x}_a \tau_{,tt} + 2\dot{x}_a \dot{x}_c \tau_{,tc} \\ - \dot{x}_c \dot{x}_b \eta_{a,bc} - 2\dot{x}_b \eta_{a,tb} - \eta_{a,tt} = 0 \end{aligned} \quad (12)$$

where $f_{,t} = \frac{\partial f}{\partial t}$ and $f_{,c} = \frac{\partial f}{\partial x_c}$. By herding together coefficients of the terms that are cubic, quartic, and linear in x_a , and the ones independent of x_a separately, and equating each of these to zero we obtain an over determined set of partial differential equations and solve for τ and η_a .

3 Symmetry generators, classical particle

The solutions of the symmetry conditions provide us the generators of the group. In this section, we explicitly obtain the generators for the cases mentioned in the introduction.

For a charged particle moving in the absence of electromagnetic field the equation of motion is given by,

$$m\ddot{x}_k = 0 \quad (13)$$

and it is well known that the symmetry condition, which is equation(12), gives rise to the eight parameter symmetry generator of the general projective transformation as given by

$$\begin{aligned} \mathbf{X} = & [a_1 + a_2t + a_3x_a + a_4tx_a + a_5t^2] \frac{\partial}{\partial t} \\ & + [a_6 + a_7t + a_8x_a + a_5tx_a + a_4(x_a)^2] \frac{\partial}{\partial x_a} \end{aligned} \quad (14)$$

for each of the equations(13). We mention this result, so that the generators can be compared with the results we would obtain later for our examples.

We first give the complete analysis of a simpler case, which is the case(i) as mentioned in the introduction.

Case (i):

For the motion of a charged particle in the presence of a magnetic field proportional to the coordinate vectors, the equations of motion are

$$\ddot{x}_k = \beta \varepsilon_{kbc} \dot{x}_b x_c = \omega_k \quad (15)$$

where the above expression in the middle is the Lorentz force acting on a charged particle. The magnetic field for this case is proportional to x_c .

Substituting

$$\omega_k = \beta \varepsilon_{kbc} \dot{x}_b x_c \quad (16)$$

into equation (12) we obtain, in general, coupled partial differential equations for τ and η by equating to zero the terms corresponding to various powers of \dot{x}_l .

Consideration of the term with $\dot{x}_a \dot{x}_b \dot{x}_c$ in Eq.(12) tells us

$$\tau_{,bc} = 0. \quad (17)$$

Hence we may have

$$\tau = A_l(t)x_l + B(t) + C. \quad (18)$$

The terms quadratic in \dot{x} give

$$\begin{aligned} & -\beta \dot{x}_b \dot{x}_c x_m \tau_{,c} \varepsilon_{abm} + 2\beta \dot{x}_l \dot{x}_b q_m \tau_{,b} \varepsilon_{alm} \\ & + \beta \dot{x}_r \dot{x}_a x_s \tau_{,b} \varepsilon_{brs} + 2\dot{x}_a \dot{x}_c \tau_{,tc} - \dot{x}_c \dot{x}_b \eta_{a,bc} = 0 \end{aligned} \quad (19)$$

This shows that τ has to be independent of x_l . Hence

$$\tau = B(t) + C, \quad (20)$$

and η may have the form

$$\eta_a = D(t)x_a + E_l(t)\varepsilon_{lam}x_m + F(t) + G. \quad (21)$$

The terms linear in \dot{x}_l provide

$$\begin{aligned} & \beta \dot{x}_l \varepsilon_{alb} \eta_b + \beta \dot{x}_c x_m \varepsilon_{abm} \eta_{b,c} - \beta \dot{x}_b x_m \varepsilon_{abm} \tau_{,t} + 2\beta \dot{x}_l x_m \varepsilon_{alm} \tau_{,t} \\ & - \beta \dot{x}_r x_s \varepsilon_{brs} \eta_{a,b} - 2\dot{x}_b \eta_{a,tb} + \dot{x}_a \tau_{,tt} = 0. \end{aligned} \quad (22)$$

This demands

$$B(t) = tH + C, \quad (23)$$

and also η_a has to be independent of t , giving

$$\eta_a = -Hx_a + E_l \varepsilon_{lam} x_m. \quad (24)$$

Thus we obtain five generators

$$\mathbf{X}_a = \varepsilon_{akb} x_b \frac{\partial}{\partial x_k}, \quad \mathbf{X}_4 = \frac{\partial}{\partial t}, \quad \mathbf{X}_5 = t \frac{\partial}{\partial t} - x_a \frac{\partial}{\partial x_a} \quad (25)$$

and their Lie algebra

$$[\mathbf{X}_a, \mathbf{X}_b] = \varepsilon_{abc} \mathbf{X}_c, \quad [\mathbf{X}_a, \mathbf{X}_4] = 0, \quad [\mathbf{X}_a, \mathbf{X}_5] = 0, \quad [\mathbf{X}_4, \mathbf{X}_5] = \mathbf{X}_4. \quad (26)$$

A comparison with the results of similar analysis for the Kepler problem [17] shows that the first four generators are identical, the first three corresponding to the generators of the three dimensional rotation group and \mathbf{X}_4 is the generator for time translation. However in this case the law corresponding to Kepler's third law goes instead like

$$\tilde{t}\tilde{r} = tr = constant. \quad (27)$$

The Lie algebra represented by equation (26) corresponds, in the notation of Stephani [17], to the group $SO(3) \times G_2IIa$, where the G_2IIa is a group with the two generators

$$\mathbf{X}_4 = \frac{\partial}{\partial t}, \quad \mathbf{X}_5 = t \frac{\partial}{\partial t} - x_a \frac{\partial}{\partial x_a}. \quad (28)$$

We can find their extension from the formula

$$\dot{\eta}_a = \frac{d\eta_a}{dt} - \dot{x}_a \frac{d\tau}{dt} \quad (29)$$

and obtain, denoting the extensions by $\dot{\mathbf{X}}$,

$$\dot{\mathbf{X}}_a = \varepsilon_{akb} (x_b \frac{\partial}{\partial x_k} + \dot{x}_b \frac{\partial}{\partial \dot{x}_k}), \quad (30)$$

$$\dot{\mathbf{X}}_4 = \frac{\partial}{\partial t}, \quad (31)$$

$$\dot{\mathbf{X}}_5 = t \frac{\partial}{\partial t} - x_a \frac{\partial}{\partial x_a} - 2\dot{x}_a \frac{\partial}{\partial \dot{x}_a}, \quad \text{scaling}. \quad (32)$$

Henceforth we scale β , which is a function of the coupling constant, mass, etc. to one.

Through an analysis that is similar to the above procedure, we find the the generators of the symmetry groups for the following examples.

Case (ii):

If the particle is further constrained to move on the surface of a sphere of unit radius, the equation of motion becomes

$$\ddot{x}_a = \varepsilon_{abc} \dot{x}_b \dot{x}_c - x_a \dot{x}_k \dot{x}_k. \quad (33)$$

This is equivalent to the case of a particle moving in the presence of a magnetic monopole centered at the origin of the sphere. Witten has generalized this idea to arbitrary dimensions for field theoretic considerations. In the above and henceforth we have scaled β to one. As has been pointed out by Witten [33], one faces trouble in attempting to derive these equations of motion by using the usual procedure of variation of a Lagrangian since no obvious term can be included in the Lagrangian whose variation would give the equation of motion (15). Hence it would be more appropriate here to consider the group analysis of the equations of motion directly to obtain all the Lie point symmetries. Usually the Noether symmetries are a subclass of these. However, the present analysis cannot give any of the non-Lie symmetries.

With ω_a being equal to the right hand side of equation (33), the group analysis shows that there is only one trivial time translation besides the generators of the angular momentum for this problem,

$$\mathbf{X}_a = \varepsilon_{akb} x_b \frac{\partial}{\partial x_k}, \quad \mathbf{X}_4 = \frac{\partial}{\partial t}. \quad (34)$$

Same is the case if we ignore the term containing ε_{abc} in equation (33). So also for a magnetic field $\mathbf{B} = -bx_1 \mathbf{e}_1 + (B_0 + bx_3) \mathbf{e}_3$, which is an idealized version of the Stern-Gerlach magnetic field [36].

Case (iii):

If a potential like $\frac{1}{r^2}$ is only present we find the generators with extensions to be

$$\begin{aligned} \dot{\mathbf{X}}_a &= \varepsilon_{abc} \left(x_c \frac{\partial}{\partial x_b} + \dot{x}_c \frac{\partial}{\partial \dot{x}_b} \right), \quad \text{space rotations,} \\ \dot{\mathbf{X}}_4 &= \frac{\partial}{\partial t}, \quad \text{time translation,} \end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{X}}_5 &= 2t \frac{\partial}{\partial t} + x_a \frac{\partial}{\partial x_a} - \frac{1}{2} \dot{x}_a \frac{\partial}{\partial \dot{x}_a}, \\
\text{Kepler like scaling law } \frac{t}{r^2} &= \text{constant}, \\
\dot{\mathbf{X}}_6 &= t^2 \frac{\partial}{\partial t} + tx_a \frac{\partial}{\partial x_a} + x_a \frac{\partial}{\partial \dot{x}_a} - \dot{x}_a \frac{\partial}{\partial \dot{x}_a}
\end{aligned} \tag{35}$$

The vector fields have the commutation relations

$$\begin{aligned}
[\mathbf{X}_a, \mathbf{X}_b] &= \varepsilon_{abc} \mathbf{X}_c, \\
[\mathbf{X}_a, \mathbf{X}_4] &= [\mathbf{X}_a, \mathbf{X}_5] = [\mathbf{X}_a, \mathbf{X}_6] = 0 \\
[\mathbf{X}_4, \mathbf{X}_5] &= 2\mathbf{X}_4, \quad [\mathbf{X}_4, \mathbf{X}_6] = \mathbf{X}_5, \quad [\mathbf{X}_5, \mathbf{X}_6] = 2\mathbf{X}_6
\end{aligned} \tag{36}$$

The classical Kepler problem with $\frac{1}{r}$ potential has a different scaling law of $\frac{t^2}{r^3}$ and also does not possess the symmetry corresponding to generator \mathbf{X}_6 . However, it possesses a Runge-Lenz vector. Stephani has given a general method to obtain such conserved vectors in the Lagrangian formulation. In the quantum mechanical case, if the eigenvalues are taken instead the Hamiltonian operator, an enhanced symmetry with closed Lie algebra occurs for $\frac{1}{r}$ potential. For $\frac{1}{r^2}$ potential we could not find a classical Runge-Lenz vector by Stephani's method. This appears to be related to orbits being not closed in such a potential[32]. However, as has already been noted, in this case new vector fields result leading to the extra symmetries.

Case (iv):

Jackiw has considered the symmetries of equation of motion, Lagrangian, and Hamiltonian for a charged particle in the field of a magnetic monopole [3, 5]. He had discovered an extra $SO(2, 1)$ hidden symmetry by scaling and physical considerations. Leonhardt and Piwnicki have explored the theoretical possibility of obtaining the field of quantized monopoles when a classical dielectric moves in a charged capacitor [34]. Since the magnetic field due to a magnetic monopole is $B_a = \frac{x_a}{r^3}$, the equation of motion is

$$\ddot{x}_a = \varepsilon_{abc} \frac{\dot{x}_b x_c}{r^3} = \omega_a \tag{37}$$

We have taken the coupling constant, mass etc. to be unity. Lie point symmetries of these equations were obtained in[37, 38]. We get the same

generators with extension as in the case (iii) for the $\frac{1}{r^2}$ potential. Zwanzinger had considered the motion of a charged particle in the presence of monopole along with a $\frac{1}{r^2}$ potential[35]. Hence the equation of motion

$$\ddot{x}_a = \varepsilon_{abc} \frac{\dot{x}_b x_c}{r^3} + \frac{\mu^2 x_a}{mr^4} \quad (38)$$

also possesses the same above symmetry. The last terms of the equation reminds of an electric dipole potential at large distances in the one dimensional case.

Case(v):

We have obtained for the case of the field due to both a monopole and a charge, i.e. a dyon, only the first four generators of equation (35) for case(iii).

Case (vi):

But for a velocity dependent potential with equations of motions of the form

$$\ddot{x}_a = \dot{x}_a x_k x_k \quad (39)$$

we again find five symmetry generators, the first four being the same as $\dot{\mathbf{X}}_a$ and \mathbf{X}_4 while

$$\hat{\mathbf{X}}_5 = 2t \frac{\partial}{\partial t} - x_a \frac{\partial}{\partial x_a}, \quad (40)$$

and with its extension,

$$\dot{\hat{\mathbf{X}}}_5 = 2t \frac{\partial}{\partial t} - x_a \frac{\partial}{\partial x_a} - 3\dot{x}_a \frac{\partial}{\partial \dot{x}_a}. \quad (41)$$

The length and time scale in this case as

$$\tilde{t}\tilde{r}^2 = tr^2. \quad (42)$$

Case (vii):

For a charged particle moving in a model magnetic field of the form

$$\mathbf{B} = \{B_x = 0, B_y = 0, B_z = -\frac{\mathcal{B}}{x^2}\} \quad (43)$$

the equations of motion are

$$\ddot{x}_1 = -\frac{\dot{x}_2 \mathcal{B}}{x_1^2}, \quad \ddot{x}_2 = \frac{\dot{x}_1 \mathcal{B}}{x_1^2}, \quad \ddot{x}_3 = 0. \quad (44)$$

Here \mathcal{B} is a constant. This magnetic field may be obtained from a current density $\mathbf{J} = (0, \frac{\mathcal{B}}{x^3}, 0)$, which is singular. It is interesting to note, however, that the Schrödinger equation can be exactly solved and corresponding energy levels be obtained in the manner of Landau [19]. The condition (12) for $a = 1$, and 2 gives

$$\tau = \lambda, \quad \eta_1 = 0, \quad \eta_2 = \sigma, \quad (45)$$

and for $a = 3$ we obtain

$$\eta_3 = \rho + x_3 \quad (46)$$

where λ , σ , and ρ are constants. This gives rise to the vector fields

$$\mathbf{X}_\tau = \lambda \frac{\partial}{\partial t}, \quad \mathbf{X}_{\eta_2} = \sigma \frac{\partial}{\partial x_2}, \quad \mathbf{X}_3 = \rho \frac{\partial}{\partial x_3}, \quad \mathbf{X}_{\eta_3} = x_3 \frac{\partial}{\partial x_3}, \quad (47)$$

that forms a solvable Lie algebra. The commutation relations are given by

$$\begin{aligned} [\mathbf{X}_\tau, \mathbf{X}_{\eta_2}] &= 0, & [\mathbf{X}_\tau, \mathbf{X}_3] &= 0, & [\mathbf{X}_\tau, \mathbf{X}_{\eta_3}] &= 0, \\ [\mathbf{X}_{\eta_2}, \mathbf{X}_3] &= 0, & [\mathbf{X}_{\eta_2}, \mathbf{X}_{\eta_3}] &= 0, & [\mathbf{X}_3, \mathbf{X}_{\eta_3}] &= \mathbf{X}_3. \end{aligned} \quad (48)$$

and correspond to direct products of two abelian groups and the group G_2IIb which has the generators

$$\begin{aligned} \mathbf{X}_3 &= \rho \frac{\partial}{\partial x_3} \\ \mathbf{X}_{\eta_3} &= x_3 \frac{\partial}{\partial x_3}. \end{aligned} \quad (49)$$

We also analyse the Landau problem with a constant magnetic field in the x_3 direction. The equation of motion is given by,

$$\ddot{x}_k = \beta \varepsilon_{kbc} \dot{x}_b B_c = \omega_k \quad (50)$$

where $\mathbf{B} = (0, 0, B)$, with B a constant. The vector field obtained is

$$\mathbf{X}_1 = \frac{\partial}{\partial x_1}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x_2}, \quad \mathbf{X}_3 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}, \quad \mathbf{X}_4 = \frac{\partial}{\partial t}. \quad (51)$$

with commutation relations

$$[\mathbf{X}_1, \mathbf{X}_2] = 0, \quad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_1, \quad [\mathbf{X}_a, \mathbf{X}_4] = 0. \quad (52)$$

Thus we have found the generators that specifies the corresponding symmetry groups for all our examples considered in the introduction.

4 Symmetry generators, quantum particle

The simplest one dimensional version of the equation

$$\ddot{x}_a = \frac{\mu^2 x_a}{mr^4} \quad (53)$$

possesses remarkable symmetries which were exploited by de Alfaro, Fubini, and Furlan to construct conformal quantum mechanics. Here x is considered as a field in zero space and one time dimension. The quantum mechanical equation for the wave function u becomes, in our notation,

$$\left(-\frac{d^2}{dx^2} + \frac{g}{x^2} + \frac{x^2}{a^2} \right) u = \frac{4r}{a} u \quad (54)$$

where $\frac{\mu^2}{m}$ is replaced by g . Here a is a constant which plays a fundamental role in the theory and r is related to appropriate raising and lowering operators. This equation can be expressed in terms of differential operator realization of $su(1, 1)$ algebra[18] and was studied in detail in[1].

There has been earlier works, where it has been shown that the dynamics of a scalar particle approaching the event horizon of a blackhole is governed by an Hamiltonian with an inverse square potential[42, 43, 44, 45, 46, 47, 2]. The scalar field can be used as a probe to study the geometry in the vicinity of the horizon and its dynamics is expected to provide clues to the inherent symmetry properties of the system. The Hamiltonian of conformal quantum mechanics fits into this. This Hamiltonian also arise as a limiting case of the brickwall model describing the low energy quantum dynamics of a field in the background of a massive Schwarzschild blackhole of mass M [46, 47]. By factorizing such a Hamiltonian, Birmingham, Gupta and Sen have found the Virasoro symmetry of the system and have studied the representation of the algebra as well as the scaling properties of the time independent modes[2]. They had obtained the full Virasoro algebra by the

requirement of unitarity of the representation. The Hamiltonian operator is in the enveloping algebra. However, here we aim to find the underlying Lie point symmetry of the equation of motion of the scalar particle, viewed as a differential equation. Of course, mathematically any two linear homogeneous ordinary differential equations can be transformed to the form, where a prime denotes a differentiation with respect to x ,

$$u'' = 0. \quad (55)$$

This equation has the eight dimensional symmetry of projective transformations. But the two equations could be different from the physics point of view having different eigenvalues and eigenfunctions. Hence we would like to see explicitly what are the Lie point symmetries of the particular equation.

For the equation

$$u'' = \omega(x, u, u') \quad (56)$$

where

$$\omega(x, u, u') = -\left(\frac{C}{x^2} + \frac{D}{x} + \hat{E}\right)u(x). \quad (57)$$

the symmetry generators are obtained from the conditions given by the equation (12) which reduces in the one dimensional case to

$$\begin{aligned} &\omega(\eta_{,u} - 2\tau_{,x} - 3u'\tau_{,u}) - \omega_{,x}\tau - \omega_{,u}\eta - \omega_{,u'}[\eta_{,x} + u'(\eta_{,u} - \tau_{,x}) - u'^2\tau_{,u}] \\ &+ \eta_{,xx} + u'(2\eta_{,xu} - \tau_{,xx}) + u'^2(\eta_{,uu} - 2\tau_{,xu}) - u'^3\tau_{,uu} = 0 \end{aligned} \quad (58)$$

For the case $C = -g$, $D = \hat{E} = 0$, equating to zero the coefficients of u'^3 and u'^2 in (58) we get

$$\tau_{,uu} = 0, \quad \eta_{,uu} = 2\tau_{,xu} \quad (59)$$

which are satisfied for

$$\tau = u\alpha(x) + \beta(x), \quad \eta = u^2\alpha'(x) + u\gamma(x) + \delta(x). \quad (60)$$

Using these and equating to zero the coefficient of u' and then considering the the terms not involving u' , we find that an interesting symmetry exists only when the coupling constant g is equal to 2. For this case we obtain

$$\begin{aligned} \tau &= \frac{1}{x}Au + Fx \\ \eta &= -\frac{1}{x^2}Au^2 + Bu + \delta(x) \end{aligned} \quad (61)$$

where A, F , and B are constants and $\delta(x)$ satisfies the same equation as u does. The vector fields are

$$\mathbf{X}_1 = x \frac{d}{dx}, \quad \mathbf{X}_2 = u \frac{d}{du}, \quad \mathbf{X}_3 = \frac{1}{x} u \frac{d}{dx} - \frac{1}{x^2} u^2 \frac{d}{du} \quad (62)$$

with commutation relations

$$[\mathbf{X}_1, \mathbf{X}_2] = 0, \quad [\mathbf{X}_1, \mathbf{X}_3] = -2\mathbf{X}_3, \quad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3 \quad (63)$$

For the case, considered in [6], the relevant equation is

$$\frac{d^2 u}{dx^2} + \frac{1}{x^2} \left[\frac{1}{4} + R^2 E^2 \right] u = 0 \quad (64)$$

where E is a generic eigenvalue and $R = 2M$. Hence g corresponds to $\left[\frac{1}{4} + R^2 E^2 \right]$ in this case. However, our result shows that only when E is imaginary with $\left[\frac{1}{4} + R^2 E^2 \right] = -2$, the symmetry will show up. Further, the $\frac{1}{x}$ and $\frac{1}{x^2}$ factors in \mathbf{X}_3 makes it ill defined as $x \rightarrow 0$ similar to the L_{-n} operators of conformal field theory or the P_m operators considered by Birmingham, Gupta, and Sen[2].

5 Complete Krause symmetry

Krause has introduced the concept of the complete symmetry group of a system by specifying two extra properties in the definition of a Lie symmetry group. This requires the manifold of solutions to be a homogeneous space on which the group action takes place and the group is specific to the system with no other system admitting it.

Besides the Lie point symmetries and the contact symmetries, new types of symmetries are to be included in order to obtain the complete symmetry group[12]. For an N dimensional system, the generators of the new symmetry was defined to be

$$Y = \left[\int \xi(t, x_1, x_2, \dots, x_N) dt \right] \partial_t + \sum_{k=1}^N \eta_k(t, x_1, x_2, \dots, x_N) \partial_{x_k} \quad (65)$$

which is different from the generators of a Lie point transformation because of the appearance of the integral of ξ . This makes it a nonlocal operator.

Nucci has developed a method based on the reduction of order to derive all Lie symmetries[13]. Later Nucci and Leach have found the existence of more nonlocal symmetries. These symmetries become local on reduction of order. In this technique, for an autonomous system, one of the unknown function is taken as the new independent variable and the system is written in the modified form. Then the standard Lie group analysis of this transformed system yields the extra symmetries leading to a complete attainment of Krause symmetries. The method can be extended to include non-autonomous systems[31]. Using a different technique, it has also again being found that the three dimensional Kepler problem is completely specified by six symmetries[16].

Besides the original Kepler problem, this method has been used to analyze many problems of physics, space science, meteorology etc. which include the Kepler problem with a drag, motion in an angle dependent force[31, 48], Lorenz equation[14], Euler-Poinsot system and Kowalevsky top[15, 16], as well as relativistically spherically symmetric systems[39]. Nucci's interactive code for determination of Lie symmetries has been used to arrive at the above results[40].

We follow the method and notation of references[13, 31] to determine the Krause symmetries for our examples. In our case the system of differential equations are given by

$$\ddot{x}_k = F_k(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) \quad (66)$$

where $k = 1, 2$, and 3 and the ω_k of equation(1) equals F_k . By standard techniques[26, 27] one obtains the generators of the Lie point group for this system and a generator is written in the form

$$X = \tau(t, x_1, x_2, x_3)\partial_t + \sum_{k=1}^3 \eta_k(t, x_1, x_2, x_3)\partial_{x_k} \quad (67)$$

To treat the velocities in the same footing as the coordinates and for reduction of order, equation(66) is next made into a set of six ordinary differential equations

$$\dot{u}_k = u_{3+k} \quad (68)$$

$$\dot{u}_{3+k} = F_k(u_1, u_2, u_3, u_4, u_5, u_6) \quad (69)$$

where u_k s are related to x_k s and \dot{x}_k s. Then one of the dependent variables, u_i s, is chosen as the new independent variable y . We take $u_3 = y$. The system (68)-(69) is now converted to a set of five ordinary differential equations depending on the variable y , with

$$\frac{du_j}{dy} = \frac{u_{3+j}}{u_6}, \quad (70)$$

$$\frac{du_{3+j}}{dy} = \frac{F_j(u_1, u_2, y, u_4, u_5, u_6)}{u_6}, \quad (71)$$

$$\frac{du_6}{dy} = \frac{F_3(u_1, u_2, y, u_4, u_5, u_6)}{u_6}, \quad (72)$$

where $j = 1, 2$. Using equation (70) we obtain

$$u_{3+j} = u_6 \frac{du_j}{dy}. \quad (73)$$

This is put back in equations (71) and (72) to give the two ordinary second order equations and one first order equation for the unknowns $u_j = u_j(y)$, and $u_6 = u_6(y)$

$$u_j'' = \frac{[F_j(u_1, u_2, y, u_1', u_2', u_6) - F_3(u_1, u_2, y, u_1', u_2', u_6)u_j']}{u_6^2}, \quad (74)$$

$$u_6' = \frac{1}{u_6} F_3(u_1, u_2, y, u_1', u_2', u_6), \quad (75)$$

where a prime denotes differentiation with respect to y . For the above system we write a generator for the Lie symmetry group as

$$Z = V(y, u_1, u_2, u_6)\partial_y + \sum_{j=1}^2 G_j(y, u_1, u_2, u_6)\partial_{u_j} + G_6(y, u_1, u_2, u_6)\partial_{u_6}. \quad (76)$$

These can be transformed to the old form, Y , of the operators of equation (65) by replacing u_j, y, u_6 with x_j, x_3, \dot{x}_3 , respectively, and solving the following system of equations for ξ and η_k

$$Y(x_j) \equiv \eta_j = G_j, \quad (77)$$

$$Y(x_3) \equiv \eta_3 = V, \quad (78)$$

$$Y^{(1)} \equiv \frac{d\eta_3}{dt} - \xi \dot{x}_3 = G_6, \quad (79)$$

with $Y^{(1)}$ being the first prolongation of Y .

On application of the above technique to our example corresponding to the equation of motion (15),

$$\ddot{x}_k = \varepsilon_{kbc} \dot{x}_b x_c = \omega_k \quad (80)$$

we obtain the following set of equations.

$$\Omega_1 \equiv u_1'' = \frac{1}{u_6^2} [(u_2' y - u_2) - u_1' (u_1' u_2 - u_2' u_1)] \quad (81)$$

$$\Omega_2 \equiv u_2'' = \frac{1}{u_6^2} [(u_1 - u_1' y) - u_2' (u_1' u_2 - u_2' u_1)] \quad (82)$$

$$\Omega_6 \equiv u_6' = 1 (u_1' u_2 - u_2' u_1) \quad (83)$$

For the determination of ξ and the η 's we note that, first denoting $\eta_3 = \zeta$,

$$\frac{\partial \tilde{u}_l}{\partial \tilde{y}} = \frac{\partial u_l}{\partial y} + \epsilon [\eta_{l,y} - \zeta_{,y} u_l' + (\eta_{l,u_i} - u_l' \zeta_{,u_i}) u_i'] \quad (84)$$

and, for example, we find

$$\eta_6' = \eta_{6,y} + (\eta_{6,u_i} - \zeta_{,u_i} u_6') u_i' - \zeta_{,y} u_6' \quad (85)$$

with $u_6' = \Omega_6$. Denoting the vector-fields by $\bar{\mathbf{X}}$, the symmetry condition for our first order equation is given by

$$\bar{\mathbf{X}} \Omega_6 = \eta_{6,y} + (\eta_{6,u_i} - \zeta_{,u_i} \Omega_6) u_i' - \zeta_{,y} u_6' \quad (86)$$

where

$$\bar{\mathbf{X}} = \zeta(y, u_i) \frac{\partial}{\partial y} + \eta_j(y, u_i) \frac{\partial}{\partial u_j} \quad (87)$$

and for the second order equations the symmetry conditions take the form

$$\begin{aligned}
& \zeta \Omega_{1,y} + \eta_1 \Omega_{1,u_1} + \eta_2 \Omega_{1,u_2} + \eta_6 \Omega_{1,u_6} \\
& + [\eta_{1,y} + (\eta_{1,u_i} - \zeta_{,u_i} u_1') u_i' - \zeta_{,y} u_1'] \Omega_{1,u_1'} \\
& + [\eta_{2,y} + (\eta_{2,u_i} - \zeta_{,u_i} u_2') u_i' - \zeta_{,y} u_2'] \Omega_{1,u_2'} \\
& - [\eta_{1,u_1} - 2\zeta_{,y} - 3\zeta_{,u_i} u_i'] \Omega_1 - \eta_{1,yy} - (2\eta_{1,yu_i} - \zeta_{,yu_i}) u_i' + \zeta_{,yy} u_1' \\
& + \zeta_{,u_i} u_1' \Omega_i - \eta_{1,u_j u_i} u_i' u_j' + \zeta_{,yu_i} u_1' u_i' + \zeta_{,u_i u_j} u_1' u_i' u_j' = 0. \tag{88}
\end{aligned}$$

In the above the appropriate equations (81), (82), and (83) are to be substituted. For equations (86) and (88) to be compatible we have

$$\zeta = y, \quad \eta_1 = u_1, \quad \eta_2 = u_2 \tag{89}$$

and we obtain

$$\xi = -1 \tag{90}$$

For our example of equation (39) which is

$$\ddot{x}_a = \dot{x}_a x_k x_k \tag{91}$$

the equations for u_i 's become

$$\Omega_1 \equiv u_1'' = 0, \quad \Omega_2 \equiv u_2'' = 0, \quad \Omega_6 \equiv u_6' = u_1^2 + u_2^2 + y^2. \tag{92}$$

The compatibility of the above three equations forces ξ and η 's to be

$$\zeta = y, \quad \eta_1 = u_1, \quad \eta_2 = u_2, \quad \eta_6 = 3u_6. \tag{93}$$

Thus the vector field is given by,

$$\bar{\mathbf{X}} = y \frac{\partial}{\partial y} + u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + 3u_6 \frac{\partial}{\partial u_6} \tag{94}$$

which has a corresponding part in $\dot{\mathbf{X}}_5$ of equation (41). From equation (79) we obtain ξ to be

$$\xi = -2. \tag{95}$$

In the case of a charged particle in the magnetic field of a monopole, with the equation of motion given by (equation (37)),

$$\ddot{x}_a = \varepsilon_{abc} \frac{\dot{x}_b x_c}{r^3} = \omega_a$$

we get

$$\Omega_1 \equiv u_1'' = \frac{1}{u_6} (u_2 - u_2' y - u_1'^2 u_2 - u_1 u_1' u_2') \quad (96)$$

$$\Omega_2 \equiv u_2'' = \frac{1}{u_6} (u_1' y - u_1 + u_1' u_2 u_2' - u_1) \quad (97)$$

$$\Omega_6 \equiv u_6' = u_1 u_2' - u_2 u_1' \quad (98)$$

The symmetry conditions are satisfied for

$$\zeta = y, \quad \eta_1 = u_1, \quad \eta_2 = u_2, \quad \eta_6 = -u_6 \quad (99)$$

and this gives rise to the vector field

$$\bar{\mathbf{X}} = y \frac{\partial}{\partial y} + u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} - u_6 \frac{\partial}{\partial u_6}. \quad (100)$$

Consequently one obtains,

$$\xi = 2. \quad (101)$$

For the case of the last example, i.e, the Landau problem with a constant magnetic field, the equations of motion are

$$\ddot{x}_k = \varepsilon_{klm} \dot{x}_l B_m \quad (102)$$

with $\mathbf{B} = (0, 0, \frac{1}{x^2})$. After reduction of order the equations are

$$\Omega_1 \equiv u_1'' = -\frac{\mathcal{B} u_2''}{u_6 u_1}, \quad \Omega_2 \equiv u_2'' = -\frac{\mathcal{B} u_1'}{u_6 u_1^2}, \quad \Omega_6 \equiv u_6' = 0. \quad (103)$$

The analysis results in

$$\zeta = y, \quad \eta_1 = u_1, \quad \eta_6 = -u_6. \quad (104)$$

So we get, also for this case,

$$\xi = 2. \quad (105)$$

This gives us the vector fields

$$\mathbf{v}_1 = y \frac{\partial}{\partial y}, \quad \mathbf{v}_2 = u_1 \frac{\partial}{\partial u_1}, \quad \mathbf{v}_3 = -u_6 \frac{\partial}{\partial u_6}, \quad (106)$$

So we get back the vector field \mathbf{X}_{η_3} in \mathbf{v}_1 , but there are now two extra vector fields \mathbf{v}_2 and \mathbf{v}_3 , which are to be included in the complete symmetry group.

Thus we have found the generators that specifies the corresponding symmetry groups for all our examples as well as the generators of the complete symmetry of Krause. In our examples, we find that η_k 's do not depend on \dot{x}_N and ξ is a constant, and hence Z can be transformed into a generator of a Lie point symmetry group.

6 Conclusion

As has already been explicitly mentioned, the equations of motion of a free particle admit eight symmetries for each of the x_a s. This is the maximum number of symmetries for an ordinary second order differential equation. By including different x_a, \dot{x}_a dependent terms in the equations we do explicitly see which generators survive as symmetries and we have found corresponding complete Lie algebras. We have chosen some cases motivated by problems from physics. The original motivation of including Wess-Zumino terms in the Lagrangian has been to reduce some of its symmetries[33], and here we find that the equations of motion now support the three dimensional rotations and a time translation symmetry instead of the six vector fields as for the monopole problem without any constraint. For the other examples considered here, we get some interesting result in the form of Kepler's scaling law and the full structure of the inherent symmetry group. This we get without solving the equations of motion. In the cases where a Lagrangian could be set up, those generators operating on the Lagrangian giving zero include all the Noether symmetries[17].

It is also expected that related group analysis may provide useful information when terms are modified in the Lagrangian, due to quantum corrections, for example. We have shown explicitly how many and which generators remain as symmetries. These symmetries correspond to the transformations of

the solutions. Similar analysis for specific cases of some nonlinear equations arising out of linear equations have also been carried out in [53]. We expect that these will ultimately lead to a better understanding of the spontaneous symmetry breaking, as well.

We have also carried out the analysis using Nucci's method of reduction of order to find the complete Krause symmetries in four of our examples. It is found that in our last two examples the generators of the complete symmetry group can be transformed into a generator of a Lie point symmetry group.

For the case of a scalar particle probing the near horizon structure of a blackhole, under certain limits the Hamiltonian contains $\frac{1}{r}$ and $\frac{1}{r^2}$ potentials. It is found that there exist a symmetry with three generators only for specific constant value of the coefficient of the $\frac{1}{r^2}$ term and in the absence of $\frac{1}{r}$ term.

The quantum mechanical problem of a charged particle in the presence of even a constant magnetic field has many interesting mathematical structures [51] and under certain limits can make space coordinates noncommutative [54]. Klishevich and Plyuschay have found a universal algebraic structure at the quantum level for the two dimensional case in the presence of certain magnetic fields[51]. Nonlinear superconformal symmetry of the fermion-monopole system has been extensively studied in[52]. It would be a motivation to seek the existence of analogous structures for our cases. It would be interesting to study these quantum aspects as well as the non-Abelian quantum kinematics in the framework of group theoretic quantization programme of Krause [55] for above nonuniform magnetic fields.

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